

1. **(15%)** A copper wire of cross-sectional area $2.00 \times 10^{-6} \text{ m}^2$ and length 4.00 m has a current of 2.00 A uniformly distributed across that area. (a) **(5%)** What is the magnitude of the electric field along the wire? (b) **(10%)** How much electrical energy is transferred to thermal energy in 30 min? (Resistivity of copper is $1.69 \times 10^{-8} \Omega \cdot \text{m}$)

%%%

Solution:

(a) $\therefore \vec{E} = \rho \vec{J}$

$$\therefore |\vec{E}| = \rho |\vec{J}| = (1.69 \times 10^{-8} \Omega \cdot \text{m}) \left(\frac{2.00 \text{ A}}{2.00 \times 10^{-6} \text{ m}^2} \right) = 1.69 \times 10^{-2} \text{ V/m (N/C)}$$

(5%)

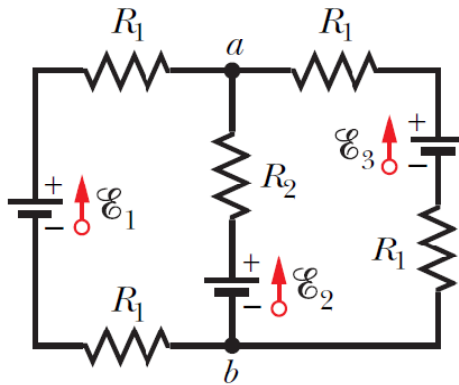
(b) $\therefore R = \rho \frac{L}{A} = (1.69 \times 10^{-8} \Omega \cdot \text{m}) \left(\frac{4 \text{ m}}{2.00 \times 10^{-6} \text{ m}^2} \right) = 0.0338 \Omega$. **(5%)**

$$\therefore P = i^2 R = (2.00 \text{ A})^2 (0.0338 \Omega) = 0.135 \text{ W}.$$

Assuming a steady rate, the thermal energy generated in 30 minutes is

$$(0.135 \text{ J/s})(30 \times 60 \text{ s}) = 2.43 \times 10^2 \text{ J}.$$
 (5%)

2. (15%) In Fig.1, the resistances are $R_1 = 1.0 \Omega$ and $R_2 = 2.0 \Omega$, and the ideal batteries have emfs $\xi_1 = 2.0 \text{ V}$ and $\xi_2 = \xi_3 = 4.0 \text{ V}$. What are the (a) magnitude and (b) direction (up or down) of the current in battery 1, the (c) magnitude and (d) direction of the current in battery 2, and the (e) What is the potential difference $V_a - V_b$? (每小題各 3 分)



%%

Solution:

(a) We note that the R_1 resistors occur in series pairs, contributing net resistance $2R_1$ in each branch where they appear. Since $\xi_2 = \xi_3$ and $R_2 = 2R_1$, from symmetry we know that the currents through ξ_2 and ξ_3 are the same: $i_2 = i_3 = i$. Therefore, the current through ξ_1 is $i_1 = 2i$. Then from $V_b - V_a = \xi_2 - iR_2 = \xi_1 + (2R_1)(2i)$ we get

$$i = \frac{\xi_2 - \xi_1}{4R_1 + R_2} = \frac{4.0 \text{ V} - 2.0 \text{ V}}{4(1.0 \Omega) + 2.0 \Omega} = 0.33 \text{ A}.$$

Therefore, the current through ξ_1 is $i_1 = 2i = 0.67 \text{ A}$.

(b) The direction of i_1 is downward.

(c) The current through ξ_2 is $i_2 = 0.33 \text{ A}$.

(d) The direction of i_2 is upward.

(e) $V_a - V_b = -iR_2 + \xi_2 = -(0.333 \text{ A})(2.0 \Omega) + 4.0 \text{ V} = 3.3 \text{ V}$.

3. (5%) Switch S in Figure is closed at time $t = 0$, to begin charging an initially uncharged capacitor of capacitance $C = 20.0 \mu\text{F}$ through a resistor of resistance $R = 20.0 \Omega$. At what time is the potential across the capacitor equal to half of the potential across the resistor?

%%

Solution:

The potential across the capacitor equals to half of the potential across the resistor

$$\therefore \frac{1}{2}V_R = V_C$$

$$\therefore V_R = 2V_C \dots \dots \dots (1)$$

Charging a Capacitor

$$\therefore q = C\varepsilon \left(1 - e^{\frac{-t}{RC}}\right)$$

$$\therefore V_C = \frac{q}{C} = \varepsilon \left(1 - e^{\frac{-t}{RC}}\right) \dots \dots \dots (2)$$

$$\therefore i = \frac{\varepsilon}{R} e^{\frac{-t}{RC}}$$

$$\therefore V_R = iR = \varepsilon e^{\frac{-t}{RC}} \dots \dots \dots (3)$$

(2)(3)代入(1)

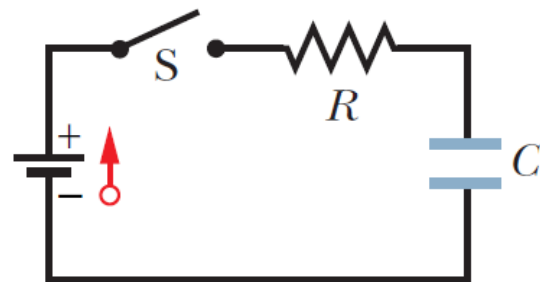
$$V_R = 2V_C \Rightarrow \varepsilon e^{\frac{-t}{RC}} = 2\varepsilon \left(1 - e^{\frac{-t}{RC}}\right)$$

$$\Rightarrow e^{\frac{-t}{RC}} = 2 \left(1 - e^{\frac{-t}{RC}}\right) \Rightarrow e^{\frac{-t}{RC}} = 2 - 2e^{\frac{-t}{RC}}$$

$$\Rightarrow 3e^{\frac{-t}{RC}} = 2 \Rightarrow e^{\frac{-t}{RC}} = \frac{2}{3}$$

$$\Rightarrow \frac{-t}{RC} = \ln \frac{2}{3} \Rightarrow \frac{t}{RC} = \ln \frac{3}{2}$$

$$\Rightarrow t = RC \ln \frac{3}{2} = (20 \Omega)(20 \mu\text{F})(0.405) = 1.62 \times 10^{-4} \text{ s}$$



4. **(15%)** A singly charged ion of mass m is accelerated from rest by a potential difference ΔV . It is then deflected by a uniform magnetic field (perpendicular to the ion's velocity) into a semicircle of radius R . Now a doubly charged ion of mass m' is accelerated through the same potential difference and deflected by the same magnetic field into a semicircle of radius $R' = 2R$. What is the ratio of the masses of the ions?

%%%%%%%%%

Solution:

$$\frac{1}{2}mv^2 = q(\Delta V) \Rightarrow v = \sqrt{\frac{2q(\Delta V)}{m}} \text{ (5\%)}$$

$$r = \frac{mv}{qB} = \frac{m\sqrt{\frac{2q(\Delta V)}{m}}}{qB} \text{ (5\%)}$$

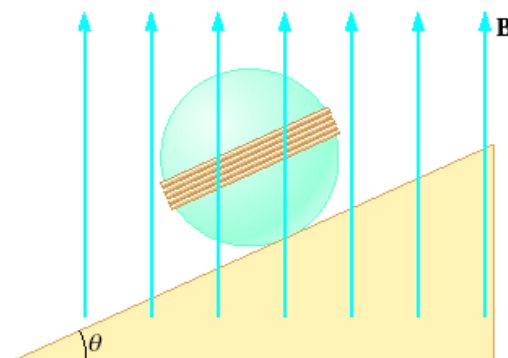
$$r^2 = \frac{m}{q} \cdot \frac{2(\Delta V)}{B^2} \quad \text{and} \quad (r')^2 = \frac{m'}{q'} \cdot \frac{2(\Delta V)}{B^2}$$

$$m = \frac{qB^2r^2}{2(\Delta V)} \quad \text{and} \quad m' = \frac{q'B^2(r')^2}{2(\Delta V)}$$

$$\frac{m'}{m} = \frac{q'}{q} \cdot \frac{(r')^2}{r^2} = \left(\frac{2e}{e}\right) \left(\frac{2R}{R}\right)^2 = 8 \text{ (5\%)}$$

%%%%%%%%%

5. (15%) A nonconducting sphere has mass 80.0 g and radius 20.0 cm. A flat compact coil of wire with 5 turns is wrapped tightly around it, with each turn concentric with the sphere. As shown in Fig.3, the sphere is placed on an inclined plane that slopes downward to the left, making an angle θ with the horizontal, so that the coil is parallel to the inclined plane. A uniform magnetic field of 0.350 T vertically upward exists in the region of the sphere. What current in the coil will enable the sphere to rest in equilibrium on the inclined plane?



%%%%%%%%%

Solution: P29.53

The sphere is in translational equilibrium, thus

$$f_s - Mg \sin \theta = 0 \quad (3\%) \quad (1)$$

The sphere is in rotational equilibrium. If torques are taken about the center of the sphere, the magnetic field produces a clockwise torque of magnitude $\mu B \sin \theta$, and the frictional force a counterclockwise torque of magnitude $f_s R$, where R is the radius of the sphere. Thus:

$$f_s R - \mu B \sin \theta = 0 \quad (8\%) \quad (2)$$

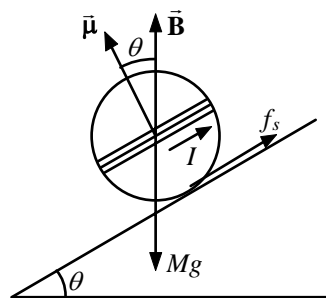
From (1): $f_s - Mg \sin \theta = 0$. Substituting this in (2) and canceling out $\sin \theta$, one obtains

$$\mu B = MgR \quad (3)$$

Now $\mu = NI\pi R^2$. (2%) Thus (3) gives $I = \frac{Mg}{\pi NBR} \quad (2\%) = \frac{(0.08 \text{ kg})(9.80 \text{ m/s}^2)}{\pi(5)(0.350 \text{ T})(0.2 \text{ m})} =$

0.713 A.

The current must be counterclockwise as seen from above.

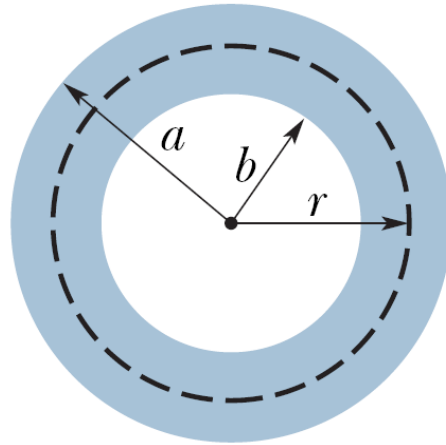


6. (15%) Fig. 6 shows a cross section of a hollow cylindrical conductor of radii a and b , carrying a uniformly distributed current i . (a) (8%) Show that the magnetic field magnitude $B(r)$ for the radial distance r in the range $b < r < a$ is given by

$$B = \frac{\mu i}{2\pi(a^2 - b^2)} \frac{r^2 - b^2}{r}$$

- (b) (7%) Show that when $r = a$, this equation gives the magnetic field magnitude B at the surface of a long straight wire carrying current i ; when $r = b$, it gives zero magnetic field; and when $b = 0$, it gives the magnetic field inside a solid conductor of radius a carrying current i .

Fig. 6.



Solution:

- (a) For the circular path L of radius r concentric with the conductor

$$\oint_L \vec{B} \cdot d\vec{s} = 2\pi r B = \mu_0 i_{enc} = \mu_0 i \frac{\pi(r^2 - b^2)}{\pi(a^2 - b^2)}$$

Thus, $B = \frac{\mu_0 i}{2\pi r} \frac{(r^2 - b^2)}{a^2 - b^2}$ (8%)

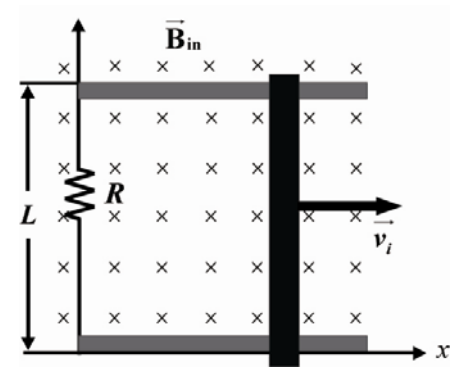
- (b) At $r = a$, the magnetic field strength is

$$\frac{\mu_0 i}{2\pi(a^2 - b^2)} \frac{(a^2 - b^2)}{a} = \frac{\mu_0 i}{2\pi a}$$

At $r = b$, $B \propto r^2 - b^2 = 0$. Finally, for $b = 0$

$$B = \frac{\mu_0 i}{2\pi a^2} \frac{r^2}{r} = \frac{\mu_0 i r}{2\pi a^2}$$
 (7%)

7. (15%) The conducting bar illustrated in Fig.4 moves on two frictionless, parallel rails in the presence of a uniform magnetic field direction into the page. The bar has mass m and length L . The bar is given an initial velocity v_i to the right and is released at $t = 0$. According to the Faraday's law, the velocity of the bar might change with time. Please find the velocity of the bar as a function of time.



%%%

Solution:

兩種方法：

- (a) Use Newton's laws to find the velocity as a function of time

Apply Newton's second law $F_x = ma = m \frac{dv}{dt} = -ILB$ (5%)

substitute $I = \frac{BLv}{R}$ (3%)

$$\therefore m \frac{dv}{dt} = -\frac{B^2 L^2}{R} v$$

$$\Rightarrow \frac{dv}{v} = -\left(\frac{B^2 L^2}{mR}\right) dt \Rightarrow \int_{v_i}^v \frac{dv}{v} = -\left(\frac{B^2 L^2}{mR}\right) \int_0^t dt$$
 (4%)

$$\Rightarrow \int_{v_i}^v \frac{dv}{v} = -\left(\frac{B^2 L^2}{mR}\right) \int_0^t dt \Rightarrow \ln\left(\frac{v}{v_i}\right) = -\left(\frac{B^2 L^2}{mR}\right) t$$

$$\Rightarrow v(t) = v_i e^{-\left(\frac{B^2 L^2}{mR}\right) t}$$
 (3%)

- (b) Use energy approach to find the velocity as a function of time

The power entering the resistor equals to that leaving the bar

$$I^2 R = -\frac{d}{dt} \left(\frac{1}{2} m v^2 \right)$$
 (5%)

substitute $I = \frac{BLv}{R}$ (3%) $\therefore \frac{B^2 L^2 v^2}{R} = -m v \frac{dv}{dt}$

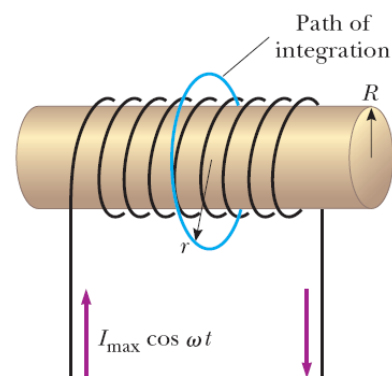
$$\Rightarrow \frac{dv}{v} = -\left(\frac{B^2 L^2}{mR}\right) dt \Rightarrow \int_{v_i}^v \frac{dv}{v} = -\left(\frac{B^2 L^2}{mR}\right) \int_0^t dt$$
 (4%)

$$\Rightarrow \ln\left(\frac{v}{v_i}\right) = -\left(\frac{B^2 L^2}{mR}\right) t$$

$$v(t) = v_i e^{-\left(\frac{B^2 L^2}{mR}\right) t}$$
 (3%)

8. (15%) In Fig. 5. A long solenoid of radius R_1 has n_1 turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_{\max} \cos \omega t$, where I_{\max} is the maximum current and ω is the angular frequency of the alternating current.

- (a) Determine the magnitude of the induced electric field **outside** the solenoid at a distance $r > R$ from its long central axis.
- (b) What is the magnitude of the induced electric field **inside** the solenoid, a distance r from its axis?



Solution:

(a) $\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$ (5%)

$\Phi_B = B\pi R^2$ (2%)

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\pi R^2) = -\pi R^2 \frac{dB}{dt}$$

$$B = \mu_0 n_1 I = \mu_0 n_1 I_{\max} \cos \omega t$$

$$-\frac{d\Phi_B}{dt} = -\pi R^2 \mu_0 n_1 I_{\max} \frac{d}{dt}(\cos \omega t) = \pi R^2 \mu_0 n_1 I_{\max} \omega \sin \omega t$$

$$\oint \vec{E} \cdot d\vec{s} = E(2\pi r) = \pi R^2 \mu_0 n_1 I_{\max} \omega \sin \omega t$$

$$E = \frac{\mu_0 n_1 I_{\max} \omega R^2}{2r} \sin \omega t$$
 (3%)

(b) $\Phi_B = B\pi r^2$ (2%)

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\pi r^2) = -\pi r^2 \frac{dB}{dt}$$

$$-\frac{d\Phi_B}{dt} = -\pi r^2 \mu_0 n_1 I_{\max} \frac{d}{dt}(\cos \omega t) = \pi r^2 \mu_0 n_1 I_{\max} \omega \sin \omega t$$

$$\Rightarrow E(2\pi r) = \pi r^2 \mu_0 n_1 I_{\max} \omega \sin \omega t$$

$$E = \frac{\mu_0 n_1 I_{\max} \omega}{2} r \sin \omega t$$
 (3%)

9. (15%) A long coaxial cable (Fig. 7) consists of two concentric cylinders with radii a and b . Its central conductor carries a uniform current I , the outer conductor providing the return path. (a) Calculate the energy stored in the magnetic field for a length ℓ of such a cable. (b) What is the self-inductance of a length ℓ of coaxial cable?

%%%%%%%%%

Solution:

藏於磁場中的磁能

$$W_B = \int_{\infty} \frac{1}{2\mu_o} B^2 d\tau \quad (3\%) \quad (1)$$

$d\tau$ 為體積基素, ∞ 代表所有空間

由 $\oint \vec{B} \cdot d\vec{s} = \mu_o I_{enc}$ 得

$$\begin{aligned} \vec{B} &= \frac{\mu_o I r}{2\pi R^2} \quad (\text{逆時針方向}) \quad (r < a) \quad (2\%) \\ &= \frac{\mu_o I}{2\pi r} \quad (\text{逆時針方向}) \quad (a < r < b) \quad (2\%) \\ &= 0 \quad (r > b) \end{aligned}$$

取長度為 ℓ 之體積基素 $d\tau = 2\pi r \ell dr$

代入(1)式

$$\begin{aligned} W_B &= \int_0^a \frac{1}{2\mu_o} \left(\frac{\mu_o I r}{2\pi a^2} \right)^2 2\pi r \ell dr + \int_a^b \frac{1}{2\mu_o} \left(\frac{\mu_o I}{2\pi r} \right)^2 2\pi r \ell dr \\ &= \frac{\mu_o I^2 \ell}{4\pi a^4} \int_0^a r^3 dr + \frac{\mu_o I^2 \ell}{4\pi} \int_a^b \frac{1}{r} dr \\ &= \frac{\mu_o I^2 \ell}{16\pi} + \frac{\mu_o I^2 \ell}{4\pi} \ln \frac{b}{a} \quad (3\%) \quad (J) \end{aligned}$$

(a) 單位長度的磁能為

$$\frac{W_B}{\ell} = \frac{\mu_o I^2}{16\pi} + \frac{\mu_o I^2}{4\pi} \ln \frac{b}{a} \quad (J/m)$$

(b) 電纜線之自感係數

$$L = \frac{2W_B}{I^2} \quad (3\%) = \frac{\mu_o \ell}{8\pi} + \frac{\mu_o \ell}{2\pi} \ln \frac{b}{a} \quad (2\%) \quad (H)$$

單位長度的自感係數為

$$\frac{L}{\ell} = \frac{\mu_o}{8\pi} + \frac{\mu_o}{2\pi} \ln \frac{b}{a} \quad (H/m)$$