國立中山大學 98 學年度普通物理(二)第一次期中考 99.04.13

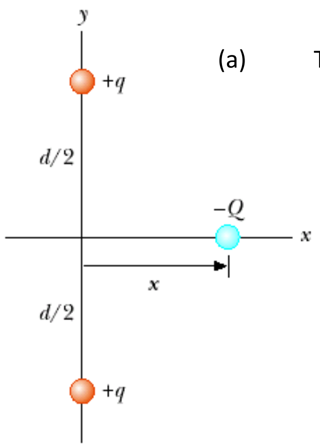
1. Two identical particles, each having charge +*q*, are fixed in space and separated by a distance *d*. A third point charge –*Q* is free to move and lies initially at rest on the perpendicular bisector of the two fixed charges a distance *x* from the midpoint between the two fixed charges. (a) Show that if *x* is small compared with *d*, the motion of –*Q* will be simple harmonic along the perpendicular bisector. Determine the period of that motion. (5%) (b) How fast will the charge –*Q* be moving when it is at the midpoint between the two fixed charges, if initially it is released at a distance *a* << *d* from the midpoint? (5%)

%%%

Ans:

The top charge exerts a force on the negative charge $\frac{k_e qQ}{\left(\frac{d}{2}\right)^2 + x^2}$ which is directed upward and to the left, at an angle of $\tan^{-1}\left(\frac{d}{2x}\right)$ to the *x*-axis. The two positive charges together exert force

$$\left(\frac{2k_e qQ}{\left(\frac{d^2}{4}+x^2\right)}\right)\left(\frac{(-x)\,\hat{\mathrm{i}}}{\left(\frac{d^2}{4}+x^2\right)^{1/2}}\right) = m\mathbf{a} \quad \text{or for} \quad x << \frac{d}{2}, \quad \mathbf{a} \approx \frac{-2k_e qQ}{m d^3/8}\mathbf{x}.$$



(a) The acceleration is equal to a negative constant times the excursion from equilibrium, as in $\mathbf{a} = -\omega^2 \mathbf{x}$, so we have Simple Harmonic Motion with $\omega^2 = \frac{16k_e qQ}{m d^3}$.

$$T = \frac{2\pi}{\omega} = \boxed{\frac{\pi}{2}\sqrt{\frac{m d^3}{k_e qQ}}}$$, where *m* is the mass of the object with charge $-Q$.

(b) $$v_{max} = \omega A = \boxed{4a\sqrt{\frac{k_e qQ}{m d^3}}}$$

2. An electric dipole in a uniform electric field is displaced slightly from its equilibrium position, as shown in the following figure, where ϑ is small. The separation of the charges is $2a$, and the moment of inertia of the dipole is *I*. Assuming the dipole is released from this position.

(a) Calculate the magnitude of the torque on the dipole. (3%)

(b)Show that its angular orientation exhibits simple harmonic motion. (4%)

(c) Find the frequency of the simple harmonic motion. (3%)

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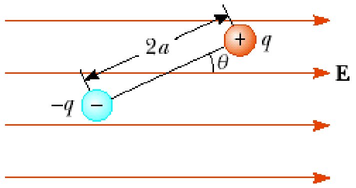
Ans: The electrostatic forces exerted on the two charges result in a net torque $\tau = -2Fa\sin\theta = -2Eqa\sin\theta$.

For small θ, $\sin\theta \approx \theta$ and using $p = 2qa$, we have $\tau = -Ep\theta$.

The torque produces an angular acceleration given by $\tau = I\alpha = I\dfrac{d^2\theta}{dt^2}$



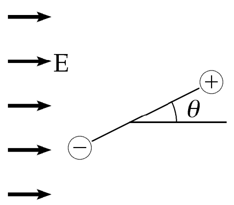


Fig. 2

Combining these two expressions for torque, we have $\dfrac{d^2\theta}{dt^2} + \left(\dfrac{Ep}{I}\right)\theta = 0$

This equation can be written in the form $\dfrac{d^2\theta}{dt^2} = -\omega^2\theta$ where $\omega^2 = \dfrac{Ep}{I}$.

This is the same form as Equation 15.5 and the frequency of oscillation is found by comparison with Equation 15.11, or $f = \dfrac{1}{2\pi}\sqrt{\dfrac{pE}{I}} = \boxed{\dfrac{1}{2\pi}\sqrt{\dfrac{2qaE}{I}}}$.

3. In Fig. 3, positive charge +q is spread uniformly along a thin nonconducting rod of length d. Derive the magnitude and direction (relative to the positive direction of the x axis) of the electric field produced at point P, at distance d from the rod along its perpendicular bisector? (8%)

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$$dq = \lambda dx$$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(x^2+d^2)} \quad (1\%)$$

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(x^2+d^2)}\frac{d}{(x^2+d^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\cdot d\cdot dx}{(x^2+d^2)^{3/2}} \quad (1\%)$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\int_{-d/2}^{d/2}\frac{\lambda}{(x^2+d^2)^{1/2}}\cdot\frac{d}{(x^2+d^2)}dx = \frac{\lambda d}{4\pi\varepsilon_0}\int_{-d/2}^{d/2}\frac{1}{(x^2+d^2)^{3/2}}dx \quad (2\%)$$

$$= \frac{\lambda d}{4\pi\varepsilon_0}\frac{x}{d^2(x^2+d^2)^{1/2}}\Bigg|_{-d/2}^{d/2} \quad 2\%$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{(5/4)^{1/2}d} = \frac{\lambda}{4\pi\varepsilon_0}\frac{2}{(5)^{1/2}d} \quad (2\%)$$

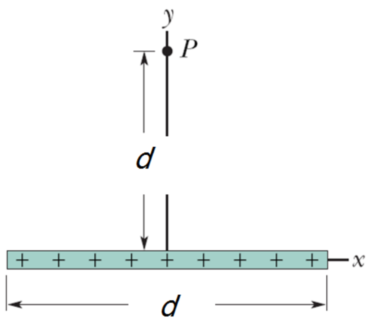

Fig. 3

4. Positive charge +4q is spread uniformly along a thin nonconducting square loop of length d at each edge, (Fig. 4). Find the magnitude and direction of the electric field produced at point P2, at distance d on top of edge of square loop?

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$$E = \frac{\lambda}{4\pi\varepsilon_0}\frac{2}{(5)^{1/2}d}$$

$$E_z = \frac{\lambda}{4\pi\varepsilon_0}\frac{2}{(5)^{1/2}d}\cdot\cos\theta\cdot 4 = \frac{\lambda}{4\pi\varepsilon_0}\frac{2}{(5)^{1/2}d}\cdot\frac{\sqrt{3}}{2}\cdot 4 = \frac{\lambda}{4\pi\varepsilon_0}\frac{4}{d}\cdot\sqrt{\frac{3}{5}} \quad (2\%)$$

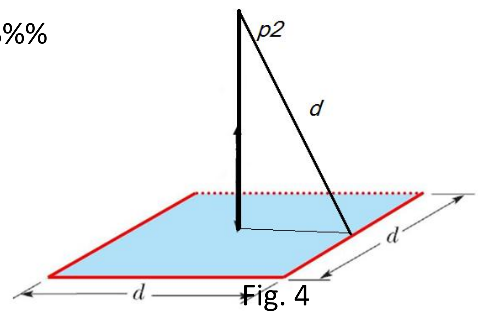

Fig. 4

5. How much work is required to turn an electric dipole 180° in a uniform electric field of magnitude E = 45.0 N/C if p = 3.0 × 10^{25} C • m and the initial angle is 45°? (3%)

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$$U = -\vec{p}\cdot\vec{E} = -pE\cos\theta \quad (1\%)$$

$$U_i = -\vec{p}\cdot\vec{E} = -pE\cos 45°$$

$$U_f = -\vec{p}\cdot\vec{E} = -pE\cos 225°$$

$$W = \Delta U = -pE(\cos 225° - \cos 45°) = \sqrt{2}pE = \sqrt{2}\cdot 3.0\times 10^{25}\cdot 45 = 1.9\cdot 10^{27} \quad (2\%)$$

6. A solid **insulating** sphere of radius a carriers a net negative charge -*Q* **uniformly distributed** throughout its volume. A conducting spherical hollow shell of inner radius *b* and outer radius *c* is concentric with the solid sphere and carriers a net charge 2*Q*. (a) Using Gauss's law, find the electric field in the regions where the radius *r* satisfies (i) *r* < *a*, (ii) *a* < *r* < *b*, (iii) *b* < *r* < *c*, (iv) *r* > *c*. (12%) (b) Determine the accumulated (累積) charge per unit area on the inner and outer surfaces of the hollow sphere (shell). (4%) (c) What will change in (a) if the inner solid sphere becomes conducting with the same carrying negative charge -*Q*? (4%)

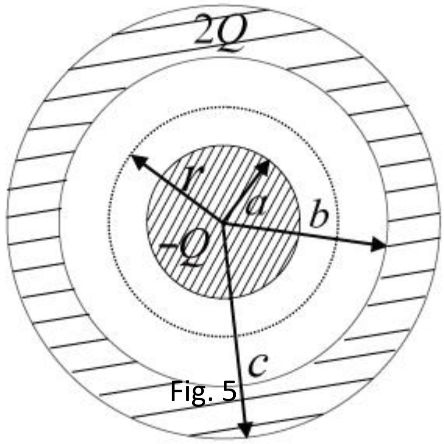

Fig. 5

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(a) Applying Gauss's law, the students should find the Gaussian surface where electric field is constant.

(i) for *r* < *a*,

$$\oint EdA = E\oint dA = E(4\pi r^2) = \frac{-Q}{\epsilon_0}\frac{\frac{4\pi}{3}r^3}{\frac{4\pi}{3}a^3} = \frac{-Q}{\epsilon_0}\frac{r^3}{a^3},$$

$$E = -\frac{1}{4\pi\epsilon_0}\frac{Q}{a^3}r(\text{or } -k_e\frac{Q}{a^3}r)$$

(ii) for *a* < *r* < *b*,

$$\oint EdA = E\oint dA = E(4\pi r^2) = \frac{-Q}{\epsilon_0},$$

$$E = -\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}(\text{or } -k_e\frac{Q}{r^2})$$

(iii) for *b* < *r* < *c*, the electric field must be zero inside the conducting spherical shell.

$$E = 0.$$

(iv) for *r* > *c*,

$$\oint EdA = E\oint dA = E(4\pi r^2) = \frac{2Q-Q}{\epsilon_0} = \frac{Q}{\epsilon_0},$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}(\text{or } k_e\frac{Q}{r^2})$$

[每一小題可以列出 Gauss's Law 給 1 分，算出正確結果給滿分 (3 分)]

(b) In the inter surface, the accumulated charge is *Q* (such that electric field inside conducting shell becomes zero, by Gauss's law). The accumulated charge per unit area is $Q/4\pi b^2$.
In the outer surface, the accumulated charge is also *Q* (2*Q* – *Q*). The charge density is $Q/4\pi c^2$.

[每一小題各給 2 分]

(c) The only change is in the region of *r* < *a*, where *E* = 0 instead.

[答案對給 4 分]

7. Charge Q distributed uniformly on a thin ring of radius R. (See Fig. 6)

(a) Find the electric potential V at a point P on the central axis of the ring and a distance z from the center. Expressed V in terms of Q, R, and z. (4%)

(b) Using the result of part (a) derive the expression of electric field $\vec{E}$ at point P. (4%)

(c) When an electron is put on the central axis very near the center of the ring($z << R$) with zero initial velocity, show that the motion of the electron is simple harmonic motion, and find the period of the motion. (4%)

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(a) $$V = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{r} \qquad (1)$$

在右圖線圈頂端取一電荷基素 dq . R 爲 dq 至 P 點的距離. 由於電荷全部分佈於圓形線圈上, 故 r 爲定值, (1)式可寫成以下形式

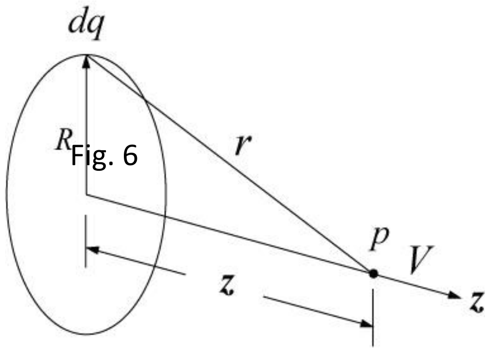

Fig. 6

$$V = \frac{1}{4\pi\varepsilon_o r}\oint dq = \frac{Q}{4\pi\varepsilon_o r} \qquad (2)$$

將 $r = \sqrt{z^2 + R^2}$ 代入(2)式得 P 點的電位爲

$$V = \frac{Q}{4\pi\varepsilon_o\sqrt{z^2 + R^2}}$$

(b) $$E_z = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{Q}{4\pi\varepsilon_o\sqrt{z^2 + R^2}}\right) = -\frac{Qz}{4\pi\varepsilon_o(z^2 + R^2)^{\frac{3}{2}}}$$

$$E_x = -\frac{\partial V}{\partial x} = 0 \quad , \qquad E_y = -\frac{\partial V}{\partial y} = 0$$

$\therefore$ P 點的電場爲 $\vec{E} = \hat{z}E_z = \dfrac{Qz}{4\pi\varepsilon_o(z^2 + R^2)^{\frac{3}{2}}}\hat{z}$

(c)電子靜置於軸上任意點, 釋放之瞬間所受庫倫力 $\vec{F}$ 爲

$$\vec{F} = -e\vec{E} = -\frac{eQz}{4\pi\varepsilon_o(z^2 + R^2)^{\frac{3}{2}}}\hat{z}$$

當 $z << R$ $\qquad \vec{F} \cong -\dfrac{eQ}{4\pi\varepsilon_o R^3}\vec{z} = -k\vec{z}$

式中 $\qquad k = \dfrac{eQ}{4\pi\varepsilon_o R^3}$ = 定值, 由上式得知電子做簡諧運動.

由牛頓第二定律

$$\vec{F} = -k\vec{z} = m_e \frac{d^2\vec{z}}{dt^2}$$

式中 m_e 爲電子質量

$$\frac{d^2\vec{z}}{dt^2} + \frac{k}{m_e}\vec{z} = 0$$

上式爲質點作簡諧運動之微分方程式，令 $\omega^2 = \frac{k}{m_e}$

則電子之振盪週期爲

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_e}{k}} = 2\pi\left(\frac{4\pi\varepsilon_o R^3 m_e}{eQ}\right)^{\frac{1}{2}}$$

$$= \left(\frac{16\pi^3\varepsilon_o R^3 m_e}{eQ}\right)^{\frac{1}{2}}$$

8. Charge is distributed on a linear thin line of infinitely long with a constant charge density λ. Find the electric potential *V* at a point of a perpendicular distance *R* from the line charge. (See Fig. 7) (10%)

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【解】**注意: 此題電荷之分佈區域爲無限大, 故不得用以下方法計算.**

$$V = \frac{1}{4\pi\varepsilon_o}\int\frac{dq}{r} \quad 及 \quad V = \int_r^{\infty}\vec{E}\cdot d\vec{l} \quad 或 \quad V = -\int_{\infty}^{r}\vec{E}\cdot d\vec{s}$$

先計算距線電荷 *R* 遠處 *P* 點的電場.

方法 1. 用連續分佈電荷求電場的方法

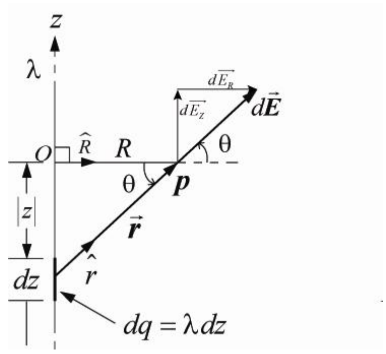


Fig. 7

由 $$d\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{dq}{r^2}\hat{\mathbf{r}}$$

取電荷基素 $dq = \lambda dz$ (見上圖), *dq* 沿 *z*-軸變換位置時, *d***E** 的方向 $\hat{\mathbf{r}}$ 也隨著改變. 故須將 $d\vec{E}$ 分成 *z*-與 *R*-分量,即 $d\vec{E} = \hat{\mathbf{R}}dE_R + \hat{\mathbf{z}}dE_z$

$$\vec{E} = \hat{\mathbf{R}}E_R + \hat{\mathbf{z}}E_z = \vec{E}_R + \vec{E}_z$$

$$\vec{E}_R = \hat{\mathbf{R}}\int dE_R = \hat{\mathbf{R}}\int\cos\theta dE = \hat{\mathbf{R}}\frac{\lambda}{4\pi\varepsilon_o}\int_{-\infty}^{+\infty}\frac{\cos\theta dz}{r^2}$$

$$r = R\sec\theta \qquad |z| = -z = R\tan\theta$$

$$dz = -R\sec^2\theta d\theta$$

將 *r* 及 *dz* 值代入積分號內

$$\vec{E}_R = \hat{\mathbf{R}}\frac{\lambda}{4\pi\varepsilon_o}\int_{-\infty}^{+\infty}\frac{\cos\theta(-R\sec^2\theta d\theta)}{(R\sec\theta)^2}$$

$$= \hat{\mathbf{R}}\frac{\lambda}{4\pi\varepsilon_o R}\int_{\pi/2}^{-\pi/2}(-\cos\theta)d\theta$$

$$= -\hat{\mathbf{R}}\frac{\lambda}{4\pi\varepsilon_o R}\left[\sin(-\frac{\pi}{2}) - \sin(\frac{\pi}{2})\right] = \hat{\mathbf{R}}\frac{\lambda}{2\pi\varepsilon_o R}$$

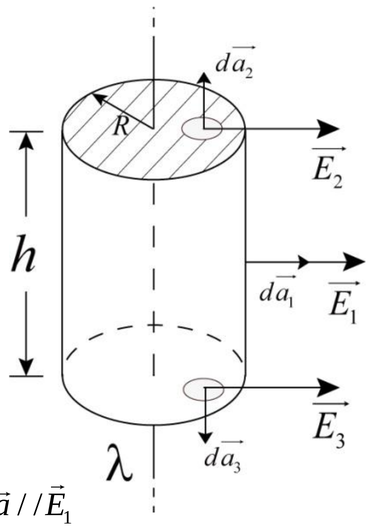
同理可得

$$\vec{E}_z = \hat{\mathbf{z}}\int dE_z = \hat{\mathbf{z}}\int \sin\theta dE = \hat{\mathbf{z}}\frac{\lambda}{4\pi\varepsilon_o}\int_{-\infty}^{+\infty}\frac{\sin\theta dz}{r^2}$$

$$= \hat{\mathbf{z}}\frac{\lambda}{4\pi\varepsilon_o R}\left[\cos(-\frac{\pi}{2}) - \cos(\frac{\pi}{2})\right] = 0$$

$$\therefore \qquad \vec{E} = \hat{\mathbf{R}}E_R + \hat{\mathbf{z}}E_z = \hat{\mathbf{R}}\frac{\lambda}{2\pi\varepsilon_o R}$$

方法 2. 用高斯定律求電場的方法



$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_o}$$

取半徑爲 *R*，高爲 *h*，軸在線電荷上的圓柱面爲高斯面如右圖所示．由電荷的對稱分佈得知：

彎曲面上：

$E_1 = E =$ 常數，　$\vec{E}_1 = \hat{R}E$，面積基素 $d\vec{a} // \vec{E}_1$

上底：　$\vec{E}_2 \perp d\vec{a}_2$；　　　下底 ：$\vec{E}_3 \perp d\vec{a}_3$

$$\oint \vec{E}\cdot d\vec{a} = \int_{彎曲面}\vec{E}_1\cdot d\vec{a}_1 + \int_{上底}\vec{E}_2\cdot d\vec{a}_2 + \int_{下底}\vec{E}_3\cdot d\vec{a}_3$$

$$= \int_{彎曲面} E\,da + 0 + 0 = E2\pi$$

$$\frac{q_{enc}}{\varepsilon_o} = \frac{1}{\varepsilon_o}(\lambda h)$$

代入高斯定律得　$E2\pi Rh = \frac{\lambda h}{\varepsilon_o}$，　$E = \frac{\lambda}{2\pi\varepsilon_o R}$

P 點的電場爲　$\vec{E} = \frac{\lambda}{2\pi\varepsilon_o R}\hat{R}$

再由電場求電位差(見右圖)

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{s}$$

$$= \frac{\lambda}{2\pi\varepsilon_o}\int_a^b \frac{1}{R}\hat{R}\cdot d\vec{s}$$

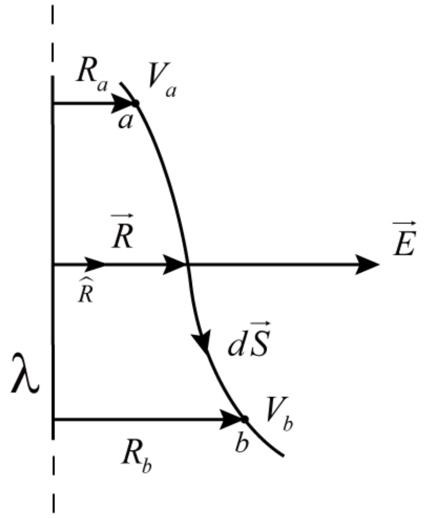
$$= \frac{\lambda}{2\pi\varepsilon_o}\int_{R_a}^{R_b} \frac{1}{R}dR$$

$$= \frac{\lambda}{2\pi\varepsilon_o}\ln\frac{R_b}{R_a}$$

設定 $R_b = R_o,\ V_b = 0$； $R_a = R,\ V_a = V$

得 P 點的電位為

$$V = \frac{\lambda}{2\pi\varepsilon_o}\ln\frac{R_o}{R}$$



9. In Fig. 8, a potential difference V = 100 V is applied across a capacitor arrangement with capacitances C_1 = 10.0 F, C_2 = 5.00 F, and C_3 = 15.0 F. What are (a) charge q_3, (b) potential difference V_3, and (c) stored energy U_3 for capacitor 3, (d) q_1, (e) V_1, and (f) U_1 for capacitor 1, and (g) q_2, (h) V_2, and (i) U_2 for capacitor 2? (10%)

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Solution:

1. (a) The potential difference across C_1 (the same as across C_2) is given by

$$V_1 = V_2 = \frac{C_3 V}{C_1 + C_2 + C_3} = \frac{(15.0\,\mu\text{F})(100\,\text{V})}{10.0\,\mu\text{F}+5.00\,\mu\text{F}+15.0\,\mu\text{F}} = 50.0\,\text{V}.$$

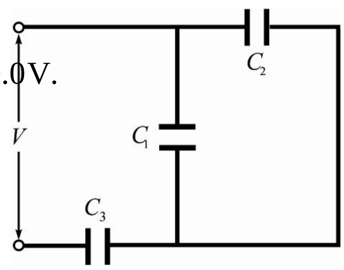

Fig. 8

Also, $V_3 = V - V_1 = V - V_2$ = 100 V – 50.0 V = 50.0 V. Thus,

$$q_1 = C_1 V_1 = (10.0\,\mu\text{F})(50.0\,\text{V}) = 5.00\times10^{-4}\,\text{C}$$
$$q_2 = C_2 V_2 = (5.00\,\mu\text{F})(50.0\,\text{V}) = 2.50\times10^{-4}\,\text{C} \qquad (2\%)$$
$$q_3 = q_1 + q_2 = 5.00\times10^{-4}\,\text{C} + 2.50\times10^{-4}\,\text{C} = 7.50\times10^{-4}\,\text{C}.$$

(b) The potential difference V_3 was found in the course of solving for the charges in part (a). Its value is V_3 = 50.0 V.(1%)

(c) The energy stored in C_3 is $U_3 = C_3 V_3^2/2 = (15.0\,\mu\text{F})(50.0\,\text{V})^2/2 = 1.88\times10^{-2}\,\text{J}$. (1%)

(d) From part (a), we have $q_1 = 5.00\times10^{-4}\,\text{C}$, and (1%)

(e) V_1 = 50.0 V, as shown in (a).(1%)

(f) The energy stored in C_1 is

$$U_1 = \frac{1}{2} C_1 V_1^2 = \frac{1}{2}(10.0\,\mu\text{F})(50.0\,\text{V})^2 = 1.25\times10^{-2}\,\text{J}. \; (1\%)$$

(g) Again, from part (a), $q_2 = 2.50\times10^{-4}\,\text{C}$.(1%)

(h) V_2 = 50.0 V, as shown in (a).(1%)

(i) The energy stored in C_2 is $U_2 = \frac{1}{2} C_2 V_2^2 = \frac{1}{2}(5.00\,\mu\text{F})(50.0\,\text{V})^2 = 6.25\times10^{-3}\,\text{J}$. (1%)

10. A parallel-plate capacitor has plates of area 0.12 m^2 and a separation of 1.2 cm. A battery charges the plates to a potential difference of 120 V and is then disconnected. A dielectric slab of thickness 4.0 mm and dielectric constant 4.8 is then placed symmetrically between the plates. (a) What is the capacitance before the slab is inserted? (b) What is the capacitance with the slab in place? What is the free charge *q* (c) before and (d) after the slab is inserted? What is the magnitude of the electric field (e) in the space between the plates and dielectric and (f) in the dielectric itself? (g) With the slab in place, what is the potential difference across the plates? (h) How much external work is involved in inserting the slab? (10%)

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Solution:

(a) Initially, the capacitance is

$$C_0 = \frac{\varepsilon_0 A}{d} = \frac{\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)(0.12\ \mathrm{m^2})}{1.2\times10^{-2}\ \mathrm{m}} = 89\ \mathrm{pF}. \quad (1\%)$$

(b) Working through Sample Problem 25-7 (**Halliday, page 674**) algebraically, we find:

$$C = \frac{\varepsilon_0 A\kappa}{\kappa(d-b)+b} = \frac{\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)(0.12\,\mathrm{m^2})(4.8)}{(4.8)(1.2-0.40)(10^{-2}\,\mathrm{m})+(4.0\times10^{-3}\,\mathrm{m})} = 1.2\times10^{2}\,\mathrm{pF}. \quad (2\%)$$

(c) Before the insertion, $q = C_0V$ (89 pF)(120 V) = 11 nC. (1%)

(d) Since the battery is disconnected, *q* will remain the same after the insertion of the slab, with *q* = 11 nC. (1%)

(e) $E = q/\varepsilon_0 A = 11\times10^{-9}\ \mathrm{C}/(8.85\times10^{-12}\ \tfrac{\mathrm{C^2}}{\mathrm{N\cdot m^2}})(0.12\ \mathrm{m^2}) = 10\ \mathrm{kV/m}.$ (1%)

(f) $E' = E/$ = (10 kV/m)/4.8 = 2.1 kV/m. (1%)

(g) The potential difference across the plates is

$V = E(d-b) + E'b$ = (10 kV/m)(0.012 m – 0.0040 m)+ (2.1 kV/m)(0.0040 m) = 88 V.

(1%)

(h) The work done is

$$W_{\mathrm{ext}} = \Delta U = \frac{q^2}{2}\left(\frac{1}{C}-\frac{1}{C_0}\right) = \frac{(11\times10^{-9}\,\mathrm{C})^2}{2}\left(\frac{1}{89\times10^{-12}\,\mathrm{F}}-\frac{1}{120\times10^{-12}\,\mathrm{F}}\right) = -1.7\times10^{-7}\,\mathrm{J}.$$

(2%)

11. (a)Two sphere have radii a and b, and their centers have a distance d apart .Find the Capacitance of this system (assume $d \gg a,\ d \gg b$) (5%)

(b)A parallel-plate capacitor of plate area A is filled with two dielectrics as in Fig.9 and Fig.10, Find the capacitance in both cases. (5%)

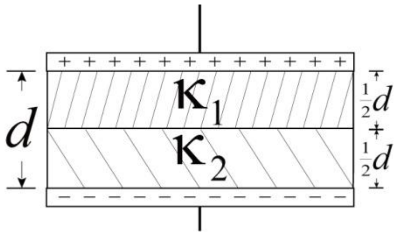

Fig. 9

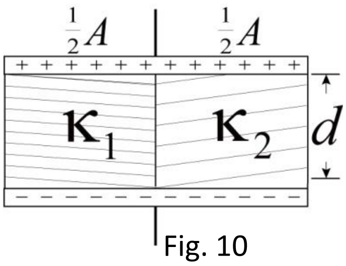

Fig. 10

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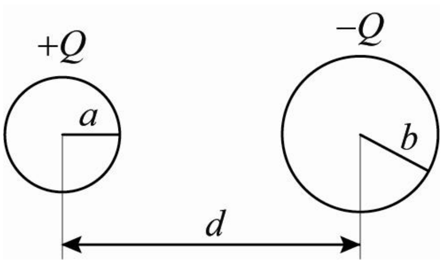
(a)

$$V_a \simeq \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} - \frac{1}{4\pi\varepsilon_0}\frac{Q}{d} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$V_b \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{-Q}{b}\right) + \frac{1}{4\pi\varepsilon_0}\frac{Q}{d} \ldots\ldots\ldots\ldots(2)$$



$$(1)\text{-}(2)\text{得}\ (a)\ V_a - V_b = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a}+\frac{1}{b}-\frac{2}{d}\right]$$

$$\therefore C = \frac{Q}{V_a - V_b} = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a}+\frac{1}{b}-\frac{2}{d}\right]}$$

(b)

$$(i)\ in\ Fig\ 1,\ C_1 = \frac{\kappa_1\varepsilon_0 A}{d/2}, C_2 = \frac{\kappa_2\varepsilon_0 A}{d/2}$$

$$\therefore \frac{1}{C} = \frac{1}{C_1} - \frac{1}{C_2} = \frac{d}{2\kappa_1\varepsilon_0 A} + \frac{d}{2\kappa_2\varepsilon_0 A} = \frac{d}{2\varepsilon_0 A}\left(\frac{1}{\kappa_1}+\frac{1}{\kappa_2}\right) = \frac{d}{2\varepsilon_0 A}\left(\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2}\right)$$

$$\therefore C = \frac{2\varepsilon_0 A}{d}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$

$$(ii)\ in\ Fig\ 2,\ C_1 = \frac{\kappa_1\varepsilon_0\left(\frac{A}{2}\right)}{d}, C_2 = \frac{\kappa_2\varepsilon_0\left(\frac{A}{2}\right)}{d}$$

$$\therefore C = C_1 + C_2 = \frac{\varepsilon_0 A}{d}\left(\frac{\kappa_1+\kappa_2}{2}\right)$$

12. Charge q is uniformly distributed in a non-conducting sphere of radius R. calculate the elecfric potential energy of the sphere. (10%)

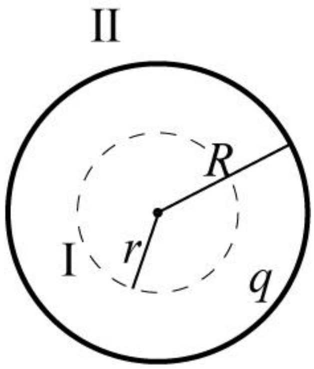
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

$$U=\frac{1}{2}\varepsilon_0\int_0^R E_1{}^2dV+\frac{1}{2}\varepsilon_0\int_R^\infty E_2{}^2dV=U_1+U_2$$

其中 E_1 為 $r \ll R$ (在球內之電場), E_2 為 $r \gg R$ (在球外之電場)

dV 為體積基素, $4\pi r^2 dr$

當 $r \ll R$ 由 $Gauss'law$



$$\oint \vec{E_0}\bullet d\vec{A}=\frac{1}{\varepsilon_0}\left(q\frac{r^3}{R^3}\right) \quad \therefore E_1\bullet 4\pi r^2=\frac{1}{\varepsilon_0}\left(q\frac{r^3}{R^3}\right)$$

$$\therefore E_1=\left(\frac{q}{4\pi\varepsilon_0 R^3}\right)r$$ (1%)

$$U_1=\frac{1}{2}\varepsilon_0\int_0^R E_1{}^2dV=\frac{1}{2}\varepsilon_0\int_0^R\left(\frac{q}{4\pi\varepsilon_0 R^3}r\right)^2\left(4\pi r^2dr\right)$$

$$=U_1=\frac{1}{2}\frac{(4\pi\varepsilon_0)q^2}{(4\pi\varepsilon_0)^2R^6}\int_0^R r^4dr=\frac{1}{2}\left(\frac{q^2}{4\pi\varepsilon_0 R^6}\right)\left[\frac{1}{5}R^5\right]=\frac{1}{40\pi\varepsilon_0}\frac{q^2}{R}$$ (4%)

$$當\ r \ll R,\ E_2=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$ (1%)

$$\therefore U_2=\frac{1}{2}\varepsilon_0\int_R^\infty E_2{}^2dV=\frac{1}{2}\varepsilon_0\int_R^\infty\left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2\left(4\pi r^2dr\right)$$

$$==\frac{q^2}{8\pi\varepsilon_0}\int_R^\infty\frac{1}{r^2}dr=\frac{q^2}{8\pi\varepsilon_0 R}$$ (3%)

$$\underline{\therefore U=\frac{q^2}{40\pi\varepsilon_0 R}+\frac{q^2}{8\pi\varepsilon_0 R}=\frac{3q^2}{20\pi\varepsilon_0 R}}$$ (1%)